COLLEGE ENTRANCE REQUIREMENTS IN MATHEMATICS.

A Preliminary Report: The National Committee on Mathematical Requirements.

General Considerations.—The committee aims to formulate standard minimum requirements adapted to the needs of academic and engineering colleges, and other institutions of similar grade. Such further requirements as may be appropriate to particular colleges or classes of colleges have been discussed in an earlier report, dealing with elective high school mathematics.

The primary purpose of college entrance requirements is to test the candidate's ability to benefit by college instruction. This ability depends—as far as our present inquiry is concerned—on (1) general intelligence, intellectual maturity and mental power; (2) specific knowledge and training required as preparation for the various courses of the college curriculum.

Mathematical ability appears to be a sufficient, but not a necessary, condition for general intelligence.* For this, as well as for other reasons, it would appear that *college entrance requirements in mathematics should be formulated primarily on the basis of the special knowledge and training required for the successful study of courses which the student will take in college.*

The separation of prospective college students from the others in the early years of the secondary school is neither feasible nor desirable. It would therefore seem to be obvious that secondary school courses for the first two years can not be planned with specific reference to college entrance requirements. Fortunately there appears to be no real conflict of interest between those students who ultimately go to college and those who do not, as far as mathematics is concerned. It

* A recent investigation made by the Department of Psychology at Dartmouth College showed that all students of high rank in mathematics had a high rating on general intelligence; the converse was not true, however.

will be made clear in what follows that a course in this subject, covering from two to two and one-half years, in a standard four-year high school, so planned as to give the most valuable mathematical training which the student is capable of receiving, will provide adequate preparation for college work.

The Selection of Topics.—In this selection preparation for college courses in mathematics need not be specifically considered. Not all college students study mathematics; it is therefore reasonable to expect college departments in this subject to adjust themselves to the previous preparation of their students. Nearly all college students do, however, study one or more of the physical sciences (astronomy, physics, chemistry) and one or more of the social sciences (history, economics, political science, sociology). Entrance requirements must therefore insure adequate mathematical preparation for these subjects. Moreover, it may be assumed that adequate preparation for these two groups of subjects will be sufficient for all other subjects the preparation for which may be expected from the secondary schools.

A recent investigation made by the National Committee gives valuable information on the question under discussion. A number of college teachers, prominent in their respective fields, were asked to assign to each of the topics in the following table its value as preparation for the elementary courses in their respective subjects. Table I gives a summary of the replies, arranged in two groups—" Physical Sciences," including astronomy, physics and chemistry, and " Social Sciences," including history, economics, sociology and political science.

TABLE I.

VALUE OF TOPICS AS PREPARATION FOR ELEMENTARY COLLEGE COURSES.

In the headings of the following table, E = essential, C = of considerable value, S = of some value, O = of little or no value, N = number of replies received. The figures in the first four columns of each group are percents of the number of replies received.

	Physical Sciences					Social Sciences				
	E	C	S	O	N	E	C	S	O	N
Negative numbers—their meaning and use	79	5	10	5	39	45	17	22	17	18
Imaginary numbers—their meaning and use	23	21	25	31	39	13	13	37	37	16
Simple formulas—their meaning and use	93	5	2		41	47	26	21	5	19

	Physical Sciences					Social Sciences				
	E	C	S	O	N	E	C	S	O	N
Graphic representation of statistical data	57	25	15	3	40	57	24	14	5	21
Graphs (mathematical and empirical):										
(*a*) as a method of representing dependence	62	16	22		37	15	54	15	15	13
(*b*) as a method of solving problems	46	20	28	6	25	18	18	46	18	11
The linear function, $y = mx + b$	78	14	8		37	29	29	14	29	14
The quadratic function, $y = ax^2 + bx + c$	59	21	17	3	34	8	8	33	50	12
Equations: Problems leading to—										
Linear equations in one unknown	98	2			41	40	7	20	33	15
Quadratic equations in one unknown	78	15	5	2	40	31	8	8	54	13
Simultaneous linear equations in two unknowns	71	24	3	3	38	33	8		58	12
Simultaneous linear equations in more than two unknowns	43	29	23	6	35	8	8	17	67	12
One quadratic and one linear equation in two unknowns	40	24	27	9	33		9	9	82	11
Two quadratic equations in two unknowns	31	19	28	22	32		9		91	11
Equations of higher degree than the 2d	10	32	32	26	31			9	91	11
Literal equations (other than formulas)	43	18	32	7	28		10	40	50	10
Ratio and proportion	84	8	3	5	39	37	26	32	5	19
Variation	50	13	20	17	30	17	33	25	25	12
Numerical computation:										
With approximate data—rational use of significant figures	61	36		3	39	40	27	20	13	12
Short-cut methods	27	38	24	10	37	29	35	23	12	17
Use of logarithms	62	29	7	2	42	12	29	29	29	17
Use of other tables to facilitate computation	24	45	26	5	38	18	29	41	12	17
Use of slide rule	24	39	26	12	38	11	39	28	22	18
Theory of exponents	36	31	25	8	36		21	21	57	14
Theory of logarithms	34	26	21	18	38	7	13	20	60	15
Arithmetic progression	16	32	38	13	37	23	29	12	35	17
Geometric progression	19	27	40	14	37	23	25	18	35	17
Binomial theorem	35	32	19	13	37	13	20	27	40	15
Probability	9	32	41	19	32	20	35	35	10	20
Statistics:										
Meaning and use of elementary concepts	23	28	31	17	29	55	36	5	5	22
Frequency distributions and frequency curves	15	19	35	32	26	47	33	10	10	21
Correlation	11	18	39	32	28	33	47	14	5	21
Numerical Trigonometry:										
Use of sine, cosine, and tangent in the solution of simple problems involving right triangles	68	21	3	8	38			25	75	12

	Physical Sciences					Social Sciences				
	E	C	S	O	N	E	C	S	O	N
Demonstrative geometry	68	15	12	6	34	·	21	43	36	14
Plane trigonometry (usual course)	57	27	11	5	37	8	23	31	38	13
Analytic geometry:										
Fundamental conceptions and methods in the plane	32	45	19	3	31		15	38	46	13
Systematic treatment of—										
Straight line	34	37	20	9	35	9	9	18	64	11
Circle	29	43	20	9	35		18	9	73	11
Conic sections	18	41	26	15	34		9	18	73	11
Polar coordinates	18	26	41	15	34			18	82	11
Empirical curves and fitting curves to observations	12	38	38	12	34	8		25	67	12

TABLE II.

Topics in Order of Value as Preparation for Elementary College Courses.

The figures in the column headed " E " are taken from Table I, taking in each case the higher of the two " E " ratings there given. The column headed " E + C " gives in each case the sum of the two ratings for " E " and " C." An asterisk indicates that the topic in question is now included in the definitions of the College Entrance Examination Board.*

	E	E + C
*Linear equations in one unknown	98	100
Simple formulas—their meaning and use	93	98
*Ratio and proportion	84	92
*Negative numbers—their meaning and use	79	84
*Quadratic equations in one unknown	78	93
The linear function: $y = mx + b$	78	92
*Simultaneous linear equations in two unknowns	71	95
Numerical trigonometry—the use of the sine, cosine, and tangent in the solution of simple problems involving right triangles	68	89
*Demonstrative geometry	68	83
Use of logarithms in computation	62	91
Computation with approximate data—rational use of significant figures	61	97
*Graphs as a method of representing dependence	61	78
The quadratic function: $y = ax^2 + bx + c$	59	80
Plane trigonometry—usual course	57	84

* The list includes all of the requirements of the College Entrance Examination Board except those relating to algebraic technique. The topic of "Negative numbers" has also been given an asterisk as it is clearly implied though not explicitly mentioned in the C. E. E. B. definitions.

	E	E + C
Graphic representation of statistical data	57	82
Statistics—meaning and use of elementary concepts	55	91
Variation	50	63
Statistics—frequency distributions and curves	47	80
*Graphic solution of problems	46	66
*Simultaneous linear equations in more than two unknowns	43	72
*Literal equations	43	61
*Simultaneous equations, one quadratic, one linear	40	64
*Theory of exponents	36	67
*Binomial theorem	35	67
Analytic geometry of the straight line	34	71
Theory of logarithms	34	60
Statistics—correlation	33	80
*Simultaneous quadratic equations	31	50
Analytic geometry of the circle	29	72
Short-cut methods of computation	29	65
Use of tables in computation (other than logarithms)	24	69
Use of slide rule	24	63
*Arithmetic progression	23	52
*Geometric progression	23	48
Imaginary numbers	23	44
Probability	20	55
Conic sections	18	59
Polar coordinates	18	44
Empirical curves and fitting curves to observations	12	50
Equations of higher degree than the second	10	42

The high value attached to the following topics is significant:

Simple formulas—their meaning and use.
The linear and quadratic functions and variation.
Numerical trigonometry.
The use of logarithms and other topics relating to numerical computation.
Statistics.

These all stand well above such standard requirements as—

Arithmetic and geometric progression.
Binomial theorem.
Theory of exponents.
Simultaneous equations involving one or two quadratic equations.
Literal equations.

These results would seem to indicate that a modification of present requirements is desirable from the point of view of college teachers in departments other than mathematics. It is interesting to note how closely the modifications suggested by this inquiry correspond to the modifications in secondary school mathematics foreshadowed by the study of the needs of the high school pupil irrespective of his possible future college attendance. The preliminary reports of the National Committee on "The Reorganization of the First Courses in Secondary School Mathematics" and on "Junior High School Mathematics" recommend that functional relationship be made the "underlying principle of the course," that the meaning and use of simple formulas be emphasized, that more attention than hitherto be given to numerical computation (especially to the methods relating to approximate data), and that work on numerical trigonometry and statistics be included. These recommendations have received widespread approval throughout the country. That they should be in such close accord with the desires of college teachers as to entrance requirements is striking. We find here the justification for the belief expressed earlier in this report that there is no real conflict between the needs of students who ultimately go to college and those who do not.

The Attitude of the Colleges.—Mathematical instruction in this country is at present in a period of transition. While a considerable number of our most progressive schools have for several years given courses embodying most of the recommendations contained in the reorganization report of the National Committee above referred to, the vast majority of schools are still continuing the older type of courses or are only just beginning to introduce modifications. The movement toward reorganization is strong, however, throughout the country, not only in the standard four-year high schools, but also in the newer junior high schools.

During this period of transition it should be the policy of the colleges, while exerting a desirable steadying influence, to help the movement toward a sane reorganization. In particular, they should take care not to place obstacles in the way of changes which are clearly in the interest of more effective

college preparation, as well as of better general education. College entrance requirements will continue to exert a powerful influence on secondary school teaching. Unless they reflect the spirit of sound progressive tendencies, they will constitute a serious obstacle.

In the present report revised definitions of plane geometry and elementary algebra are presented. So far as plane geometry is concerned, the problem of definition is comparatively simple. The proposed definition of the requirement in plane geometry does not differ from the one now in effect under the College Entrance Examination Board. A list of propositions and constructions has, however, been prepared and is included for the guidance of teachers and examiners.

In elementary algebra a certain amount of flexibility is obviously necessary, both on account of the quantitative differences among colleges and of the special conditions attending a period of transition. The former differences are recognized by the proposal of a minor and a major requirement in elementary algebra. The second of these includes the first and is intended to correspond with the two-unit rating of the College Entrance Examination Board. It includes certain optional topics which should afford the latitude needed particularly during the transition period, and should also meet the needs of colleges preferring a requirement intermediate in quantity between one unit and two.

In connection with this matter of units, the committee wishes particularly to disclaim any emphasis on a special number of years or of hours. The unit-terminology is doubtless too well established to be quite ignored in formulating college entrance requirements, but the standard definition of unit* has never been precise, and will now become much less so with the inclusion of the newer six-year program. A time allotment of four or five hours per week in the seventh grade can certainly not have the same weight as the same number of hours in the

* The following definition, formulated by the National Committee on Standards of Colleges and Secondary Schools, has been given the approval of the C. E. E. B. "A unit represents a year's study in any subject in a secondary school, constituting approximately a quarter of a full year's work. A four-year secondary school curriculum should be regarded as representing not more than sixteen units of work."

twelfth grade, and the disparity will vary with different subjects. *What is really important is the amount of subject-matter and the quality of work done in it.* The "unit" can not be anything but a crude approximation to this. The distribution of time in the school program should not be determined by any arbitrary unit scale.

As a further means of securing reasonable flexibility, the committee recommends, in case its proposed definitions should be adopted by the College Entrance Examination Board, that for a limited time—say, five years—the option be offered between examinations based on the old and on the new definitions, so far as differences between them may make this desirable.

In view of the changes taking place at the present time in mathematical courses in secondary schools and the fact that college entrance requirements should as soon as possible reflect desirable changes and assist in their adoption, the National Committee recommends that either the American Mathematical Society or the Mathematical Association of America (or both) maintain a permanent committee on College Entrance Requirements in Mathematics, such a committee to work in close coöperation with other agencies which are now or may in the future be concerned in a responsible way with the relations between colleges and secondary schools.

Definition of College Entrance Requirements.

Elementary Algebra.

Minor Requirement (one unit).

The meaning and use (including the necessary transformations) of simple formulas involving ideas with which the student is familiar and the derivation of such formulas from rules expressed in words.

The dependence of one variable upon another. Numerous illustrations and problems involving the linear function $y = mx + b$. Illustrations and problems involving the quadratic function $y = kx^2$.

The graph and graphic representations in general—their construction and interpretation—including the representation

of statistical data and the use of the graph to exhibit dependence.

Positive and negative numbers—their meaning and use.

Linear equations in one unknown quantity; their use in solving problems.

Simultaneous linear equations involving two unknown quantities; their use in solving problems.

Ratio, as a case of simple fractions; proportion, without the theorems on alternation, etc., and simple cases of variation, without the use of the symbol for variation.

The essentials of algebraic technique. This should include

- The four fundamental operations.
- Factoring of the following types:
  - Monomial factors and simple cases by grouping.
  - The difference of two squares.
  - Trinomials of the second degree (including the square of a binomial) that can be easily factored by trial.
- Fractions, including complex fractions of a simple type.
- Exponents and radicals. The laws for positive integral exponents and the meaning and use of fractional and negative exponents, but not the formal theory. The consideration of radicals may be confined to the simplification of expressions of the form $\sqrt{a^2b}$ and $\sqrt{a/b}$ and to the evaluation of simple expressions involving the radical sign. A process for extracting the square root of a number should be included, but not the process for extracting the square root of a polynomial.

Major Requirement (two units).

In addition to the minor requirement, as specified above, the following:

- Illustrations and problems involving the quadratic function $y = ax^2 + bx + c$.
- Quadratic equations in one unknown; their use in solving problems.
- The use of logarithmic tables in computation, without the formal theory.

Four of the following topics:

Numerical trigonometry—the definitions of the sine, cosine, and tangent of an angle and their use in solving problems involving right triangles. The use of three or four place tables of such functions.

Elementary statistics—including a knowledge of the fundamental concepts and simple frequency distributions, with graphic representations of various kinds.

The binomial theorem for positive integral exponents less than 8; with applications, such as compound interest.

The formula for the *n*th term and the sum of *n* terms of arithmetic and geometric progressions, with applications.

Simultaneous linear equations in three unknown quantities and simple cases of simultaneous equations involving one or two quadratic equations—their use in solving problems.

Drill in algebraic manipulation should be limited, particularly in the minor requirement, by the purpose of securing a thorough understanding of important principles and facility in carrying out those processes which are fundamental and of frequent occurrence either in common life or in the subsequent courses that a substantial proportion of the pupils will study. Skill in manipulation must be conceived of throughout as a means to an end, not as an end in itself. Within these limits, skill and accuracy in algebraic technique are of prime importance, and drill in this subject should be extended far enough to enable students to carry out the fundamentally essential processes accurately and with reasonable speed.

The consideration of literal equations, when they serve a significant purpose, such as the transformation of formulas, the derivation of a general solution (as of the quadratic equation) or the proof of a theorem, is important. As a means for drill in algebraic technique they should be used sparingly.

The solution of problems should offer opportunity throughout the course for considerable arithmetical and computational work. The conception of algebra as an extension of arithmetic should be made significant both in numerical applications and in elucidating algebraic principles. Emphasis should be placed on the use of common sense and judgment in computing from approximate data, especially with regard to

the number of significant figures retained, and on the necessity for checking the results. The use of tables to facilitate computation (such as tables of squares and square roots, interest, trigonometric functions, etc.) should be encouraged.

Plane Geometry.

One unit.

The usual theorems and constructions of good text-books, including the general properties of plane rectilinear figures; the circle and the measurement of angles; similar polygons; areas; regular polygons and the measurement of the circle.

The solution of numerous original exercises, including locus problems. Applications to the mensuration of lines and plane surfaces.

The scope of the required work in plane geometry is indicated by the List of Fundamental Propositions and Constructions, which is appended to this report. This list indicates in Section I the type of proposition which, in the opinion of the committee, may be assumed without proof or given informal treatment. Section II contains 52 propositions and 19 constructions which are regarded as so fundamental that they should constitute the common minimum of all standard courses in plane geometry. Section III gives a list of subsidiary theorems which suggests the type of additional propositions that should be included in such courses.

College Entrance Examinations.—College entrance examinations exert in many schools—and especially throughout the eastern section of the country—an influence on secondary school teaching which is very far-reaching. It is, therefore, well within the province of the National Committee to inquire whether the prevailing type of examination in mathematics serves the best interests of mathematical education and of college preparation.

The reason for the almost controlling influence of entrance examinations in the schools referred to is readily recognized. Schools sending students to Harvard, Yale, Princeton, and the larger colleges for women, or to any institution where examinations form the only or prevailing mode of admission, inevitably

direct their instruction toward the entrance examination. This remains true even if only a small percentage of the class intends to take these examinations, the point being that the success of a teacher is often measured by the success of his or her students in these examinations.

In the judgment of the committee, the prevailing type of entrance examination in algebra is primarily a test of the candidate's skill in formal algebraic manipulation. It has not been an adequate test of his understanding or of his ability to apply the principles of the subject. Moreover, it is quite generally felt that the difficulty and complexity of the formal manipulative questions, which have appeared on recent papers set by colleges and such agencies as the College Entrance Examination Board, has often been excessive. As a result, teachers preparing pupils for these examinations have inevitably been led to devote an excessive amount of time to drill in algebraic technique, without insuring an adequate understanding of the principles involved. Far from providing the desired facility, this practice has tended to impair it. For "practical skill, modes of effective technique, can be intelligently, non-mechanically used only when intelligence has played a part in their acquisition." (Dewey, "How We Think," p. 32.)

Moreover, it must be noted that authors and publishers of text-books are under strong pressure to make their content and distribution of emphasis conform to the prevailing type of entrance examination. Teachers in turn are too often unable to rise above the text-book. An improvement in the examinations in this respect will cause a corresponding improvement in text-books and in teaching.

On the other hand, the makers of entrance examinations in algebra can not be held solely responsible for the condition described. Theirs is a most difficult problem. Not only can they reply that as long as algebra is taught as it is examinations must be largely on technique,* but they can also claim with considerable force that technical facility is the only phase of algebra that can be fairly tested by an examination; that a

* The vicious circle is now complete: Algebra is taught mechanically because of the character of the entrance examination; the examination, in order to be fair, must conform to the character of the teaching.

candidate can rarely do himself justice amid unfamiliar surroundings and with a time limit ahead on questions involving real thinking in applying principles to concrete situations; that we must face here a real limitation on the power of an examination to test attainment. Many, perhaps most, teachers will agree with this claim. Past experience is on their side; no effective "power test" in mathematics has as yet been devised, and, if devised, it might not be suitable for use under conditions prevailing during an entrance examination.

But if it is true that the power of an examination is thus inevitably limited, the wisdom and fairness of using it as the sole means of admission to college is surely open to grave doubt. That many unqualified candidates are admitted under this system is not open to question. Is it not probable that many qualified candidates are at the same time excluded? If the entrance examination is a fair test of manipulative skill only, should not the colleges use additional means of learning something about the candidate's other abilities and qualifications?

Some teachers believe that an effective "power test" in mathematics is possible. Efforts to devise such a test should receive every encouragement.

In the meantime, certain desirable modifications of the prevailing type of entrance examinations are possible. The College Entrance Examination Board has recently appointed a committee to consider this question, and a conference* on this subject has been held by representatives of the College Entrance Examination Board, members of the National Committee and others. The following recommendations are taken from the report of the committee just referred to:

"Fully one third of the questions should be based on topics of such fundamental importance that they will have been thoroughly taught, carefully reviewed and deeply impressed by effective drill. They should be of such a degree of difficulty

* At this conference the following vote was unanimously passed: "Voted, That the results of examinations (of the College Entrance Examination Board), be reported by letters A, B, C, D, E and that the definition of the groups represented by these letters should be determined in each year by the distribution of ability in a standard group of papers representing widely both public and private schools.

that any pupil of regular attendance, faithful application and even moderate ability may be expected to answer them satisfactorily.

"There should be both simple and difficult questions testing the candidate's ability to apply the principles of the subject. The early ones of the easy questions should be really easy for the candidate of good average ability who can do a little thinking under the stress of an examination; but even these questions should have genuine scientific content.

"There should be a substantial question which will put the best candidates on their mettle, but which is not beyond the reach of a fair proportion of the really good candidates. This question should test the normal workings of a well-trained mind. It should be capable of being thought out in the limited time of the examination. It should be a test of the candidate's grasp and insight—not a catch question or a question of unfamiliar character making extraordinary demands on the critical powers of the candidate, or one the solution of which depends on an inspiration. Above all, this question should lie near to the heart of the subject, as all well-prepared candidates understand the subject.

"As a rule, a question should consist of a single part and be framed to test one thing—not pieced together out of several unrelated and perhaps unequally important parts.

"Each question should be a substantial test on the topic or topics which it represents. It is, however, in the nature of the case impossible that all questions be of equal value.

"Care should be used that the examination be not too long. The examiner should be content to ask questions on the important topics, so chosen that their answers will be fair to the candidate and instructive to the readers; and beyond this merely to sample the candidate's knowledge on the minor topics."

In addition, the National Committee suggests the following principles:

The examination as a whole should, as far as practicable, reflect the principle that algebraic technique is a means to an end, and not an end in itself.

Questions that require of the candidate skill in algebraic

manipulation beyond the needs of actual application should be used very sparingly.

An effort should be made to devise questions which will fairly test the candidate's understanding of principles and his ability to apply them. These should involve a minimum of manipulative complexity.

In geometry, examinations should be definitely constructed to test the candidate's ability to draw valid conclusions rather than his ability to memorize an argument.

The National Committee has published separately a report on Mathematical Terms and Symbols. It is hoped that examining bodies will avoid the use of terms and symbols not recommended by that report.

Plane Geometry.

List of Fundamental Propositions and Constructions.

I. Assumptions and Theorems for Informal Treatment.

This list contains propositions which may be assumed without proof (postulates) and theorems which it is permissible to treat informally. Some of these propositions will appear as definitions in certain methods of treatment. Moreover, teachers should feel free to require formal proofs of some, if they desire to do so. The precise wording given is not essential, nor is the order in which the propositions are here listed. The list should be taken as representative of the type of propositions which may be assumed, or treated informally, rather than as exhaustive.

1. Through two distinct points it is possible to draw one straight line, and only one.
2. A line segment may be produced to any desired length.
3. The shortest path between two points is the line segment joining them.
4. One and only one perpendicular can be drawn through a given point to a given straight line.
5. The shortest distance from a point to a line is the perpendicular distance from the point to the line.
6. From a given center and with a given radius one and only one circle can be described in a plane.

7. A straight line intersects a circle in at most two points.

8. Any figure may be moved from one place to another without changing its shape or size.

9. All right angles are equal.

10. If the sum of two adjacent angles equals a straight angle, their exterior sides form a straight line.

11. Equal angles have equal complements and equal supplements.

12. Vertical angles are equal.

13. Two lines perpendicular to the same line are parallel.

14. Through a given point not on a given straight line, one straight line, and only one, can be drawn parallel to the given line.

15. Two lines parallel to the same line are parallel to each other.

16. The area of a rectangle is equal to its base times its altitude.

II. Fundamental Theorems and Constructions.

It is recommended that theorems and constructions, other than originals, to be proved on entrance examinations be chosen from the following list. Originals and other exercises should be capable of solution by direct reference to one or more of these propositions and constructions. It should be obvious that any course in geometry capable of giving adequate training must include considerable additional material. The order here given is not intended to signify anything as to the order of presentation. It should be clearly understood that certain of the statements contain two or more theorems, and that the precise wording is not essential. The committee favors entire freedom in statement and sequence.

A. *Theorems.*

1. Two triangles are congruent if*
   (*a*) two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other;

* Teachers should feel free to separate this theorem into three distinct theorems and to use other phraseology for any such proposition. For example in 1, "Two triangles are equal if . . . ," "A triangle is determined

(*b*) two angles and a side of one are equal, respectively, to two angles and the corresponding side of the other;

(*c*) The three sides of one are equal, respectively, to the three sides of the other.

2. Two right triangles are congruent if the hypotenuse and one other side of one are equal, respectively, to the hypotenuse and another side of the other.

3. If two sides of a triangle are equal, the angles opposite these sides are equal, and conversely.*

4. The locus of a point (in a plane) equidistant from two given points is the perpendicular bisector of the line segment joining them.

5. The locus of a point equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by these lines.

6. When a transversal cuts two parallel lines, the alternate-interior angles are equal; and conversely.

7. The sum of the angles of a triangle is two right angles.

8. A parallelogram is divided into congruent triangles by either diagonal.

9. Any (convex) quadrilateral is a parallelogram

(*a*) if the opposite sides are equal;

(*b*) if two sides are equal and parallel.

10. If a series of parallel lines cut off equal segments on one transversal, they cut off equal segments on any transversal.

11. (*a*) The area of a parallelogram is equal to the base times the altitude.

(*b*) The area of a triangle is equal to one half the base times the altitude.

(*c*) The area of a trapezoid is equal to half the sum of its bases times its altitude.

by . . . ," etc. Similarly in 2, the statement might read: "Two right triangles are congruent if, beside the right angles, any two parts (not both angles) in the one are equal to corresponding parts of the other."

* It should be understood that the converse of a theorem need not be treated in connection with the theorem itself, it being sometimes better to treat it later. Furthermore a converse may occasionally be accepted as true in an elementary course if the necessity for proof is made clear, the proof to be given later.

(*d*) The area of a regular polygon is equal to half the product of its apothem and perimeter.

12. (*a*) If a straight line is drawn through two sides of a triangle parallel to the third side, it divides those sides proportionally.
    (*b*) If a line divides two sides of a triangle proportionally, it is parallel to the third side. (Proofs for commensurable cases only.)
    (*c*) The segments cut off on two transversals by a series of parallels are proportional.
13. Two triangles are similar if
    (*a*) they have two angles of one equal, respectively, to two angles of the other;
    (*b*) they have an angle of one equal to an angle of the other and the including sides are proportional;
    (*c*) their sides are respectively proportional.

14. If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.

15. The perimeters of two similar polygons have the same ratio as any two corresponding sides.

16. Polygons are similar, if they can be decomposed into triangles which are similar and similarly placed; and conversely.

17. The bisector of an (interior or exterior) angle of a triangle divides the opposite side (produced if necessary) into segments proportional to the adjacent sides.

18. The areas of two similar triangles (or polygons) are to each other as the squares of any two corresponding sides.

19. In any right triangle the perpendicular from the vertex of the right angle on the hypotenuse divides the triangle into two triangles, each similar to the given triangle.

20. In a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

21. In the same circle or in equal circles, if two arcs are equal, their central angles are equal; and conversely.

22. In any circle two angles at the center are proportional to their intercepted arcs. (Proof for commensurable case only.)

23. In the same circle or in equal circles, if two arcs are equal their chords are equal; and conversely.

24. (*a*) A diameter perpendicular to a chord bisects the chord and the arcs of the chord.
    (*b*) A diameter which bisects a chord (that is not a diameter) is perpendicular to it.

25. The tangent to a circle at a given point is perpendicular to the radius at that point; and conversely.

26. In the same circle or in equal circles, equal chords are equally distant from the center; and conversely.

27. An angle inscribed in a circle is equal to half the central angle having the same arc.

28. Angles inscribed in the same segment are equal.

29. If a circle is divided into equal arcs, the chords of these arcs form a regular inscribed polygon and tangents at the points of division form a regular circumscribed polygon.

30. The circumference of a circle is equal to $2\pi r$. (Informal proof only.)

31.* The area of a circle is equal to πr^2. (Informal proof only.)

The treatment of the mensuration of the circle should be based on related theorems concerning regular polygons, but it should be informal as to the limiting processes involved. The aim should be an understanding of the concepts involved, so far as the capacity of the pupil permits.

B. *Constructions.*

1. Bisect a line segment and draw the perpendicular bisector.
2. Bisect an angle.
3. Draw a perpendicular to a given line through a given point.
4. Construct an angle equal to a given angle.
5. Through a given point draw a straight line parallel to a given straight line.
6. Construct a triangle, given

    (*a*) the three sides;

* The total number of theorems given in this list when separated as will probably be found advantageous in teaching, this number including the converses indicated, is 52.

(*b*) two sides and the included angle;
(*c*) two angles and a side.

7. Divide a line segment into parts proportional to given segments.
8. Given an arc of a circle, find its center.
9. Circumscribe a circle about a triangle.
10. Inscribe a circle in a triangle.
11. Construct a tangent to a circle through a given point.
12. Construct the fourth proportional to three given line segments.
13. Construct the mean proportional between two given line segments.
14. Construct a triangle (polygon) similar to a given triangle (polygon).
15. Construct a triangle equal to a given polygon.
16. Inscribe a square in circle.
17. Inscribe a regular hexagon in a circle.

III. Subsidiary List of Propositions.

The following list of propositions is intended to suggest some of the additional material referred to in the introductory paragraph of Section II. It is not intended, however, to be exhaustive—indeed, the Committee feels that teachers should be allowed considerable freedom in the selection of such additional material, theorems, corollaries, originals, exercises, etc., in the hope that opportunity will thus be afforded for constructive work in the development of courses in geometry.

1. When two lines are cut by a transversal, if the corresponding angles are equal, or if the interior angles on the same side of the transversal are supplementary, the lines are parallel.
2. When a transversal cuts two parallel lines, the corresponding angles are equal, and the interior angles on the same side of the transversal are supplementary.
3. A line perpendicular to one of two parallels is perpendicular to the other also.
4. If two angles have their sides respectively parallel or respectively perpendicular to each other, they are either equal or supplementary.
5. Any exterior angle of a triangle is equal to the sum of the two opposite interior angles.

6. The sum of the angles of a convex polygon of n sides is $2(n-2)$ right angles.

7. In any parallelogram
   (*a*) the opposite sides are equal;
   (*b*) the opposite angles are equal;
   (*c*) the diagonals bisect each other.

8. Any (convex) quadrilateral is a parallelogram if
   (*a*) the opposite angles are equal;
   (*b*) the diagonals bisect each other.

9. The medians of a triangle intersect in a point which is two thirds of the distance from the vertex to the mid-point of the opposite side.

10. The altitudes of a triangle meet in a point.

11. The perpendicular bisectors of the sides of a triangle meet in a point.

12. The bisectors of the angles of a triangle meet in a point.

13. The tangents of a circle from an external point are equal.

14.* (*a*) If two sides of a triangle are unequal, the greater side has the greater angle opposite it; and conversely.
   (*b*) If two sides of one triangle are equal respectively to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second; and conversely.
   (*c*) If two chords are unequal, the greater is at the less distance from the center; and conversely.
   (*d*) The greater of two minor arcs has the greater chord; and conversely.

15. An angle inscribed in a semi-circle is a right angle.

16. Parallel lines, tangent to or cutting a circle, intercept equal arcs on the circle.

* Such inequality theorems as these are of importance in developing the notion of dependence or functionality in geometry. The fact that they are placed in our "Subsidiary List of Propositions" should not imply that they are considered of less educational value than those in List II. They are placed here because they are not "fundamental" in the same sense that the theorems of List II are fundamental.

17. An angle formed by a tangent and a chord of a circle is measured by half the intercepted arc.

18. An angle formed by two intersecting chords is measured by half the sum of the intercepted arcs.

19. An angle formed by two secants, or by two tangents, to a circle is measured by half the difference between the intercepted arcs.

20. If from a point without a circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.

21. Parallelograms, or triangles, of equal bases and altitudes are equal.

22. The perimeters of two regular polygons of the same number of sides are to each other as their radii, and also as their apothems.